Name:
PHYS 2212 Section: February 8-February 11
Recitation 03

Example

Consider an infinite wire of radius R, along the z-axis, with volume charge density $\rho(\vec{r}) = \rho_0 e^{-ar}/r$, where ρ_0 is a positive constant. Use Gauss' law to find the electric field $\vec{E}(\vec{r})$ for all space $r < R$.

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Problem

Consider a solid sphere of radius R, centered at the origin, with volume charge density $\rho(\vec{r}) = \rho_0 \cos(ar)/r^2$, where $a \equiv \pi/2R$ and ρ_0 is a positive constant. Use Gauss' law to find the electric field $\vec{E}(\vec{r})$ for all space $|\vec{r}| > R$.
Just for fun[1]: find $\vec{E}(\vec{r})$ for $|\vec{r}| < R$. How does $\vec{E}(\vec{r})$ change at $|\vec{r}| = R$?

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[1] This part of the question will have no influence on your grade.

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Example
Consider an infinite wire of radius R, along the z-axis, with volume charge density $\rho(\vec{r}) = \rho_0 e^{-ar}/r$, where ρ_0 is a positive constant. Use Gauss' law to find the electric field $\vec{E}(\vec{r})$ for all space $r < R$.

Solution:
For some volume V bounded by closed surface S, Gauss' Law can be stated as:

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} \int_V \rho dV \ , \tag{1}$$

where $d\vec{a}$ is the differential surface area element whose direction is defined to be normal to the surface and ρ is the volume charge density of the differential volume dV. The left-hand side (LHS) of Eq. 1 is the flux of the electric field through the Gaussian surface S, and the right-hand side (RHS) is the total charge enclosed by the Gaussian volume V.

Seeing that the charge distribution of the wire is independent of translations in z and rotations θ about the z-axis, one can see that the electric field lines will emanate radially from the axis of the wire (Fig. 1); that is, we can model the electric field as

$$\vec{E}(\vec{r}) = E_r(r)\hat{r} \ , \tag{2}$$

where $E_r(r)$ is an unknown function of the radial coordinate r. It follows that the flux will be easiest to calculate for the case of S being a cylinder whose axis coincides with that of the wire. Since we are interested in the electric field inside the wire, let's define the radius r' of our Gaussian cylinder to be such that $r' < R$. If the length of the Gaussian cylinder is h, then the LHS of Eq. 1 becomes:

$$\begin{aligned}
\oint_S \vec{E} \cdot d\vec{a} &= \oint_S E_r \hat{r} \cdot d\vec{a} \ , && (3)\\
&= (E_r(r'))(2\pi r' h) \ , && (4)
\end{aligned}$$

where the step from Eq. 3 to 4 is valid because the normal to the surface of the Gaussian cylinder at all points is simply $\hat{r}$ (except for the top and bottom of the cylinder, through which there is no electric field flux anyway as shown in Fig. 1).

Taking the charge density given for the wire, the RHS, or the total charge divided by ε_0, can be computed as:

$$\begin{aligned}
\frac{1}{\varepsilon_0} \int_V \rho dV &= \frac{1}{\varepsilon} \int_0^{r'} (\rho_0 \frac{e^{-ar}}{r})(2\pi r h dr) \ ,\\
&= \frac{1}{\varepsilon_0} 2\pi h \rho_0 \int_0^{r'} e^{-ar} dr \ ,\\
&= -\frac{2\pi h \rho_0}{\varepsilon_0 a} e^{-ar}\Big|_0^{r'} \ ,\\
&= \frac{2\pi h \rho_0}{\varepsilon_0 a}(1 - e^{-ar'}) \ , && (5)
\end{aligned}$$

where we've integrated over a series of infinitesimal cylinders $dV = 2\pi r h dr$.

Equations 4 and 5 can be substituted into Eq. 1 to solve for E_r:

$$\begin{aligned}
(E_r(r'))(2\pi r' h) &= \frac{2\pi h \rho_0}{\varepsilon_0 a}(1 - e^{-ar'}) \ ,\\
E_r(r') &= \frac{\rho_0(1 - e^{-ar'})}{\varepsilon_0 a r'} \ , r' < R \ .
\end{aligned}$$

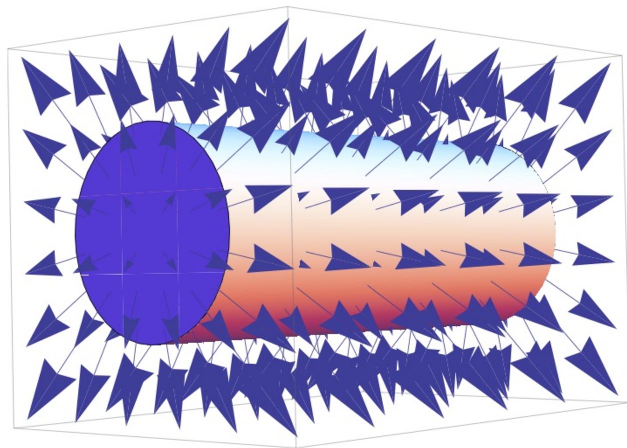

FIG. 1. Plot of the electric field flowing through the Gaussian cylindrical shell, within the larger charged wire (not shown).

Finally, making the substitution $r = r'$ and inserting E_r into Eq. 2 gives:

$$\vec{E}(\vec{r}) = \frac{\rho_0(1 - e^{-ar})}{\varepsilon_0 ar}\hat{r}\ , r < R\ .$$

Problem

Consider a solid sphere of radius R, centered at the origin, with volume charge density $\rho(\vec{r}) = \rho_0 \cos(ar)/r^2$, where $a \equiv \pi/2R$ and ρ_0 is a positive constant. Use Gauss' law to find the electric field $\vec{E}(\vec{r})$ for all space $|\vec{r}| > R$.

Just for fun: find $\vec{E}(\vec{r})$ for $|\vec{r}| < R$. How does $\vec{E}(\vec{r})$ change at $|\vec{r}| = R$?

Solution:

The important differences here from the Example are (1) we are dealing with spherical geometries rather than cylindrical ones and (2) we are interested in the electric field outside of the source as well as inside of it. Consequently, we need to use two Gaussian spheres: one that surrounds the charged sphere for the first part of the problem, and then one that fits inside of the source sphere for the second part; and we need to apply Gauss' Law separately for each Gaussian sphere.

When using the larger Gaussian sphere ($r' > R$), it is important to keep in mind that we no longer perform the volume integral over the entire Gaussian volume; because there is no charge in the space between the charged sphere and the Gaussian sphere, we must exclude that space when integrating the charge density. Thus, the volume integral must be performed over the entire volume of the charged sphere only. Proceeding to compute the LHS and RHS of Gauss' Law (Eq. 1) using the larger Gaussian sphere, we find,

LHS:

$$\oint_S \vec{E} \cdot d\vec{a} = (E_r(r'))(4\pi r'^2) ,$$

and RHS:

$$\begin{aligned}
\frac{1}{\varepsilon_0} \int_V \rho dV &= \frac{1}{\varepsilon_0} \int_0^R \left(\frac{\rho_0 \cos(ar)}{r^2} \right) (4\pi r^2 dr) , \\
&= \frac{4\pi \rho_0 (2R)}{\varepsilon_0 \pi} \sin\left(\frac{\pi R}{2R} \right) , \\
&= \frac{8\ \rho_0 R}{\varepsilon_0} .
\end{aligned}$$

where we've integrated over a series of infinitesimal shells $dV = 4\pi r^2 dr$.

And equating left and right sides results in:

$$\begin{aligned}
(E_r(r'))(4\pi r'^2) &= \frac{8\ \rho_0 R}{\varepsilon_0} , \\
E_r(r') &= \frac{2\ \rho_0 R}{\varepsilon_0 \pi r'^2} , r' > R .
\end{aligned}$$

To find the electric field inside our charged, solid sphere, we repeat the procedure using the smaller Gaussian sphere ($r' < R$). The LHS gives:

$$\begin{aligned}
\oint_S \vec{E} \cdot d\vec{a} &= \oint_S E_r \hat{r} \cdot d\vec{a} , \\
&= (E_r(r'))(4\pi r'^2) .
\end{aligned}$$

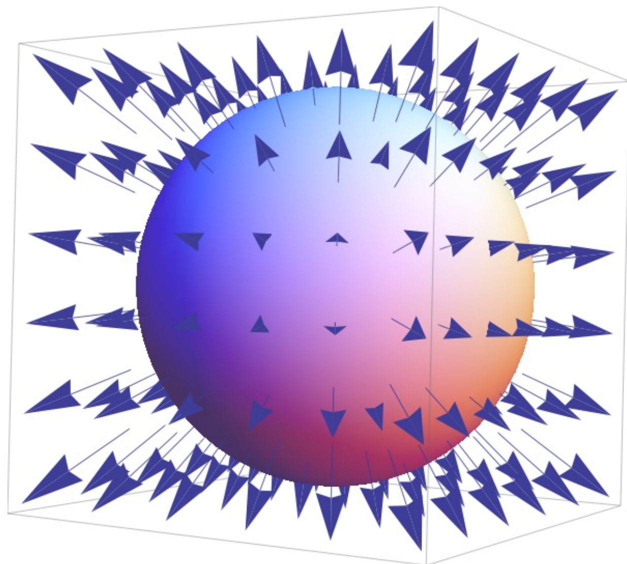

FIG. 2. Plot of the electric field through Gaussian shell.

And the RHS is:

$$\begin{aligned}
\frac{1}{\varepsilon_0}\int_V \rho dV &= \frac{1}{\varepsilon_0}\int_0^{r'}\left(\frac{\rho_0\cos(ar)}{r^2}\right)(4\pi r^2 dr)\ ,\\
&= \frac{4\pi\rho_0}{\varepsilon_0 a}\sin(ar)\bigg|_0^{r'}\ ,\\
&= \frac{4\pi\rho_0}{\varepsilon_0 a}\sin(ar')\ ,\\
&= \frac{4\pi\rho_0(2R)}{\varepsilon_0\pi}\sin\left(\frac{\pi r'}{2R}\right)\ ,\\
&= \frac{8\ \rho_0 R}{\varepsilon_0}\sin\left(\frac{\pi r'}{2R}\right)\ .
\end{aligned}$$

Equating the two sides to find E_r gives the result:

$$\begin{aligned}
(E_r(r'))(4\pi {r'}^2) &= \frac{8\ \rho_0 R}{\varepsilon_0}\sin\left(\frac{\pi r'}{2R}\right)\ ,\\
E_r(r') &= \frac{2\ \rho_0 R}{\varepsilon_0\pi {r'}^2}\sin\left(\frac{\pi r'}{2R}\right), r' < R\ .
\end{aligned}$$

Thus, the electric field at all points in space is:

$$\vec{E}(\vec{r}) = \frac{2\ \rho_0 R}{\varepsilon_0\pi r^2}\begin{cases}\sin\left(\frac{\pi r}{2R}\right)\hat{r} & ,|\vec{r}| \le R\\ \hat{r} & ,|\vec{r}| > R\end{cases}\ .$$

It's clear that the electric field is continuous at the boundary.